Large Deviations for Statistics of the Jacobi Process

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LARGE DEVIATIONS FOR STATISTICS OF JACOBI PROCESS

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Abstract. This paper is aimed to derive large deviations for statistics of Jacobi process already conjectured by M. Zani in her thesis. To proceed, we write in a simpler way the Jacobi semi-group density. Being given by a bilinear sum involving Jacobi polynomials, it differs from Hermite and Laguerre cases by the quadratic form of its eigenvalues. Our attempt relies on subordinating the process using a suitable random time-change. This will give an analogue of Mehler formula whence we can recover the desired expression by inverting some Laplace transforms. Once we did, an adaptation of Zani's result ([24]) in the non steep case will provide the required large deviations principle.

1. Introduction

The Jacobi process is a Markov process on $[-1,1]$ given by the following infinitesimal generator:

$$\mathscr{L} = (1-x^2)\frac{\partial^2}{\partial^2 x} + (px+q)\frac{\partial}{\partial x}, \qquad x \in [-1,1]$$

for some real p, q, defined up to the first time when it hits the boundary. It appears as an interest rate model in finance (see [9]) and in genetics ([11]). One of the important features is that it belongs to the class of diffusions associated to some families of orthogonal polynomials, i.e. the infinitesmal generator admits an orthogonal polynomials basis as eigenfunctions ([3]) such as Hermite, Laguerre and Jacobi polynomials . More precisely, if $P_n^{\alpha,\beta}$ denotes the Jacobi polynomial with parameters $\alpha, \beta > -1$ defined by :

$$P_n^{\alpha,\beta}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(-n, n+\alpha+\beta+1, \alpha+1; \frac{1-x}{2}\right), \qquad x \in [-1,1],$$

(see [16] for the definition of ${}_2F_1$) then we can see that :

$$\mathscr{L}P_n^{\alpha,\beta} = -n(n+\alpha+\beta+1)P_n^{\alpha,\beta}$$

for $p = -(\beta+\alpha+2)$ and $q = \beta-\alpha$. The semi group density of the process first appeared in [14] then in [22] where the author solved the forward Kolmogorov or Fokker-Planck

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equation

$$\partial_y^2[B(y)p] - \partial_y[A(y)p] = \partial_t p, \quad p = p_t(x,y),$$

where B, A are polynomials of degree 2, 1 respectively, and gave the principal solution $(p_0(x,y) = \delta_x(y))$ using the classical Sturm-Liouville theory. This gives rise to a class of stationary Markov processes satisfying :

$$\lim_{t\to\infty} p_t(x,y) = W(y) = \int_{x_1}^{x_2} W(x)p_t(x,y)dx \tag{1}$$

where W is the density function solution of the corresponding Pearson equation ([22]). In our case, p_t has the discrete spectral decomposition :

$$p_t(x,y) = \left(\sum_{n\geq 0}(R_n)^{-1}e^{-\lambda_n t}P_n^{\alpha,\beta}(x)P_n^{\alpha,\beta}(y)\right)W(y), \qquad x,y\in[-1,1] \tag{2}$$

where

$$\lambda_n = n(n+\alpha+\beta+1), \qquad W(y) = \frac{(1-y)^\alpha(1+y)^\beta}{2^{\alpha+\beta+1}B(\alpha+1,\beta+1)}$$

with B denoting the Beta function and[3] ([2], p. 99) :

$$R_n = ||P_n^{\alpha,\beta}||^2_{L^2([-1,1],W(y)dy)} = \frac{\Gamma(\alpha+\beta+2)}{2n+\alpha+\beta+1}\frac{(\alpha+1)_n(\beta+1)_n}{\Gamma(\alpha+\beta+n+1)n!}.$$

Interested in total positivity, Karlin and McGregor showed that this kernel is positive for $\alpha,\beta > -1$ ([14]). Few years later, Gasper ([13]) showed that, for $\alpha,\beta \geq -1/2$, this bilinear sum is the transition kernel of a diffusion and a solution of the heat equation governed by a Jacobi operator, generalizing a previous result of Bochner for ultraspherical polynomials ([7]). It is worth noting that λ_n has a quadratic form while in the Hermite (Brownian) and Laguerre (squared Bessel) cases $\lambda_n = n$. Hence, we will try to subordinate the Jacobi process by the mean of a random time-change in order to get a Mehler type formula. What is quite interesting is that the subordinated Jacobi process semi-group, say $q_t(x,y)$, is the Laplace transform of $p_{2/t}(x,y)$. Thus, we deduce an expression for $p_t(x,y)$ by inverting some Laplace transforms already computed by Biane, Pitman and Yor (see [5], [19]). This expression, more handable than (2), will allow us to compute the normalized cumulant generating function, and then to derive a LDP for the maximum likelihood estimate (MLE) for p in the ultraspherical case, i.e. $q = 0\,(\beta = \alpha)$, a fact conjectured by Zani in her thesis ([25]). Then, using a skew product representation of the Jacobi process involving squared Bessel processes, we construct a family $\{\hat{\nu}_t\}_t$ of estimators for the index ν of the squared Bessel process based on a Jacobi trajectory observed till time t. This satisfies a LDP with the same rate function derived for the MLE based on a squared Bessel trajectory.

[3] $(P_n^{\alpha,\beta}(x))_{n\geq 0}$ are normalized such that they form an orthogonal basis with respect to the probability measure $W(y)dy$ which is not the same used in [2].

1.1. **Inverse Gaussian subordinator.** By an *inverse Gaussian subordinator* (see [1]), we mean the process of the first hitting times of a Brownian motion with drift $B_s^{\mu} := B_s + \mu s$, $\mu > 0$, namely,

$$T_t^{\mu,\delta} = \inf\{s > 0; \quad B_s^{\mu} = \delta t\}, \qquad t,\, \delta > 0.$$

Using martingale methods, we can show that for each $t > 0$, $u \geq 0$,

$$\mathbb{E}(e^{-uT_t^{\mu,\delta}}) = e^{-t\delta(\sqrt{2u+\mu^2}-\mu)}$$

whence the density f_t of $T_t^{\mu,\delta}$ writes ([1]) :

$$f_t(s) = \frac{\delta t}{\sqrt{2\pi}} e^{\delta t\mu} s^{-3/2} \exp\left(-\frac{1}{2}(\frac{t^2\delta^2}{s} + \mu^2 s)\right) \mathbf{1}_{\{s>0\}}. \tag{3}$$

1.2. **The subordinated Jacobi Process.** Let us consider a Jacobi process $(X_t)_{t\geq 0}$. Then the semi-group of the subordinated Jacobi process $(X_{T_t^{\mu,\delta}})_{t\geq 0}$ is given by:

$$\begin{aligned} q_t(x,y) &= \int_0^{\infty} p_s(x,y) f_t(s) ds \\ &= W(y) \sum_{n\geq 0} (R_n)^{-1} \left(\int_0^{\infty} e^{-\lambda_n s} f_t(s) ds\right) P_n^{\alpha,\beta}(x) P_n^{\alpha,\beta}(y) \\ &= W(y) \sum_{n\geq 0} (R_n)^{-1} \mathbb{E}(e^{-\lambda_n T_t^{\mu,\delta}}) P_n^{\alpha,\beta}(x) P_n^{\alpha,\beta}(y). \end{aligned}$$

Writing $\lambda_n = (n+\gamma)^2 - \gamma^2$ where $\gamma = \dfrac{\alpha+\beta+1}{2}$, and substituting $\delta = 1/\sqrt{2}$, $\mu = \sqrt{2}\gamma$ for $\alpha+\beta > -1$ in the expression of f_t, one gets :

$$\mathbb{E}(e^{-\lambda_n T_t^{\mu,\delta}}) = e^{-nt}$$

so that

$$q_t(x,y) = W(y) \sum_{n\geq 0} (R_n)^{-1} e^{-nt} P_n^{\alpha,\beta}(x) P_n^{\alpha,\beta}(y).$$

The last sum has been already computed ([2], p. 385) :

$$\begin{aligned} \sum_{n=0}^{\infty} (R_n)^{-1} P_n^{\alpha,\beta}(x) P_n^{\alpha,\beta}(y) r^n &= \frac{1-r}{(1+r)^a} \sum_{m,n\geq 0} \frac{\left(\frac{a}{2}\right)_{m+n} \left(\frac{a+1}{2}\right)_{m+n}}{(\alpha+1)_m (\beta+1)_n} \frac{u^m v^n}{m!n!} \\ &= \frac{1-r}{(1+r)^a} F_4(\frac{a}{2}, \frac{a+1}{2}, \alpha+1, \beta+1; u, v) \end{aligned} \tag{4}$$

where $|r| < 1$, $a = \alpha+\beta+2$, F_4 is the Appell function ([12]) and

$$u = \frac{(1-x)(1-y)r}{(1+r)^2} \qquad v = \frac{(1+x)(1+y)r}{(1+r)^2}.$$

The integral representation of F_4 (see [12], p 51) yields :

$$q_t(x,y) = \frac{W(y)}{\Gamma(a)} \frac{1-r}{(1+r)^a} \int_0^{\infty} s^{a-1} e^{-s} {}_0F_1(\alpha+1; \frac{u}{4}s^2) {}_0F_1(\beta+1; \frac{v}{4}s^2) ds.$$

Now, from a property of the function ${}_0F_1$ (see [17], p 214)

$$ {}_0F_1(c;w(1-r)/2){}_0F_1(d;w(1+r)/2) = \sum_{n\geq 0} \frac{P_n^{\alpha,\beta}(r)}{(c)_n(d)_n} w^n,\ \alpha = c-1,\ \beta = d-1, $$

one gets :

$$ q_t(x,y) = \frac{W(y)}{\Gamma(a)} \frac{1-r}{(1+r)^a} \int_0^\infty s^{a-1} e^{-s} \sum_{n\geq 0} \frac{P_n^{\alpha,\beta}(z)}{(\alpha+1)_n(\beta+1)_n} A^n s^{2n}\, ds $$

where we set

$$ z = \frac{x+y}{1+xy}, \qquad A = \frac{(1+xy)r}{2(1+r)^2}. $$

Applying Fubini's Theorem gives :

$$ q_t(x,y) = W(y) \frac{1-r}{(1+r)^a} \sum_{n\geq 0} \frac{(a)_{2n}}{(\alpha+1)_n(\beta+1)_n} P_n^{\alpha,\beta}(z) A^n. $$

Letting $r = e^{-t}$, then

$$ q_t(x,y) = \frac{W(y) e^{\frac{a-1}{2}t}}{2^{a-1}} \frac{\sinh(t/2)}{(\cosh(t/2))^a} \sum_{n\geq 0} \frac{(a)_{2n}}{(\alpha+1)_n(\beta+1)_n} P_n^{\alpha,\beta}(z) \left[\frac{(1+xy)}{8\cosh^2(t/2)}\right]^n $$

$$ = \frac{W(y)\tanh(t/2) e^{\frac{a-1}{2}t}}{2^{a-1}} \sum_{n\geq 0} \frac{(a)_{2n}}{(\alpha+1)_n(\beta+1)_n} P_n^{\alpha,\beta}(z) \left[\frac{(1+xy)}{8}\right]^n \left(\frac{1}{\cosh(t/2)}\right)^{2n+a-1}. $$

Besides, from (3)

$$ q_t(x,y) = \frac{t\, e^{\gamma t}}{2\sqrt{\pi}} \int_0^\infty p_s(x,y)\, s^{-3/2} e^{-\gamma^2 s}\, e^{-\frac{t^2}{4s}} ds = \frac{t\, e^{\gamma t}}{2\sqrt{2\pi}} \int_0^\infty p_{2/r}(x,y)\, r^{-1/2} e^{-2\gamma^2/r}\, e^{-\frac{t^2}{8}r} dr. $$

Thus, noting that $\gamma = (a-1)/2$, we get :

$$ \int_0^\infty p_{2/r}(x,y)\, r^{-1/2} e^{-2\gamma^2/r}\, e^{-\frac{t^2}{8}r} dr = \frac{\sqrt{2\pi}\, W(y)}{2^{a-1}} \frac{\tanh(t/2)}{t/2} $$
$$ \sum_{n\geq 0} \frac{(a)_{2n}}{(\alpha+1)_n(\beta+1)_n} P_n^{\alpha,\beta}(z) \left[\frac{(1+xy)}{8}\right]^n \left(\frac{1}{\cosh(t/2)}\right)^{2n+a-1}. $$

With regard to the integrand, one easily sees that the RHS is the Laplace transform of $p_{2/r}(x,y) r^{-1/2} e^{-2\gamma^2/r}$.

1.3. **The Jacobi semi-group.** The following results are due to Biane, Pitman and Yor (see [5], [19]) :

$$ \int_0^\infty e^{-\frac{t^2}{8}s} f_{C_h}(s)\, ds = \left(\frac{1}{\cosh(t/2)}\right)^h, \qquad h > 0 \tag{5} $$

$$ \int_0^\infty e^{-\frac{t^2}{8}s} f_{T_h}(s)\, ds = \left(\frac{\tanh(t/2)}{(t/2)}\right)^h, \qquad h > 0 \tag{6} $$

where (C_h) and (T_h) are two families of Lévy processes with respective density functions f_{C_h} and f_{T_h} for fixed $h > 0$. The densities of C_h and T_1 are given by ([5]):

$$\begin{aligned} f_{C_h}(s) &= \frac{2^h}{\Gamma(h)} \sum_{p\geq 0} (-1)^p \frac{\Gamma(p+h)}{p!} f_{\tau(2p+h)}(s) \\ f_{T_1}(s) &= \sum_{k\geq 0} e^{-\frac{\pi^2}{2}(k+\frac{1}{2})^2 s} \mathbf{1}_{\{s>0\}} \end{aligned}$$

where $\tau(c) = \inf\{r > 0; B_r = c\}$ is the Lévy subordinator (the first hitting time of a standard Brownian motion B) with corresponding density :

$$f_{\tau(2p+h)}(s) = \frac{(2p+h)}{\sqrt{2\pi s^3}} \exp\left(-\frac{(2p+h)^2}{2s}\right) \mathbf{1}_{\{s>0\}}.$$

Let α, β satisfy $\alpha + \beta + 1 > 0$, thus:

$$p_{2/r}(x,y) = \frac{\sqrt{2\pi r} W(y) e^{2\gamma^2/r}}{2^{a-1}} \sum_{n\geq 0} \frac{(a)_{2n}}{(\alpha+1)_n(\beta+1)_n} P_n^{\alpha,\beta}(z) \left[\frac{(1+xy)}{8}\right]^n \times \left(f_{T_1} \star f_{C_{2n+a-1}}\right)(r)$$

or equivalently (where B stands for the Beta function):

$$p_t(x,y) = \frac{\sqrt{\pi} W(y)}{2^{\alpha+\beta}} \frac{e^{\gamma^2 t}}{\sqrt{t}} \sum_{n\geq 0} \frac{(a)_{2n}}{(\alpha+1)_n(\beta+1)_n} P_n^{\alpha,\beta}(z) \left[\frac{(1+xy)}{8}\right]^n \left(f_{T_1} \star f_{C_{2n+\alpha+\beta+1}}\right)(\frac{2}{t}).$$

1.4. **The ultraspherical case.** This case corresponds to $\alpha = \beta > -\dfrac{1}{2}$ and we will proceed in a slightly different way. Indeed, $a = 2\alpha + 2$ and

$$\begin{aligned} (4) &= \frac{1-r}{(1+r)^{2\alpha+2}} F_4(\alpha+1, \alpha+3/2, \alpha+1, \alpha+1; u, v) \\ &= \frac{1-r}{(1+r)^{2\alpha+2}} \frac{1}{(1-u-v)^{\alpha+3/2}} {}_2F_1(\frac{2\alpha+3}{4}, \frac{2\alpha+5}{4}, \alpha+1; \frac{4uv}{(1-u-v)^2}) \end{aligned}$$

where the last equality follows from (see [6])

$$F_4(b,c,b,b;u,v) = (1-u-v)^{-c} {}_2F_1(c/2, (c+1)/2, b; \frac{4uv}{(1-u-v)^2}).$$

Hence,

$$\begin{aligned} q_t(x,y) &= \frac{W(y) e^{\frac{2\alpha+1}{2}t}}{2^{\alpha+1/2}} \frac{\sinh(t)}{(\cosh t - xy)^{\alpha+3/2}} {}_2F_1(\frac{2\alpha+3}{4}, \frac{2\alpha+5}{4}, \alpha+1; \frac{(1-x^2)(1-y^2)}{(\cosh t - xy)^2}) \\ &= \frac{W(y) e^{\frac{2\alpha+1}{2}t}}{2^{\alpha+1/2}} \sinh(t) \sum_{n\geq 0} \frac{[(2\alpha+3)/4]_n [(2\alpha+5)/4]_n}{(\alpha+1)_n} \frac{[(1-x^2)(1-y^2)]^n}{(\cosh t - xy)^{2n+\alpha+3/2}}. \end{aligned}$$

Besides, for $h > 0$, we may write:

$$\left(\frac{1}{\cosh t - xy}\right)^h = \sum_{k\geq 0} \frac{(h)_k}{k!} \frac{(xy)^k}{(\cosh t)^{k+h}}$$

since $\left|\dfrac{xy}{\cosh t}\right| < 1 \quad \forall x, y \in]-1, 1[, \forall t \geq 0$ and where we used:

$$\frac{1}{(1-r)^h} = \sum_{k\geq 0} \frac{(h)_k}{k!} r^k \qquad h > 0, |r| < 1.$$

Consequently, using Gauss duplication formula,

$$q_t(x,y) = K_\alpha W(y) e^{\frac{2\alpha+1}{2}t} \tanh(t) \sum_{n,k\geq 0} \frac{\Gamma(\nu(n,k,\alpha)+1)(xy)^k}{k!\, n!\, \Gamma(\alpha+n+1)} \left[\frac{(1-x^2)(1-y^2)}{4}\right]^n \left(\frac{1}{\cosh t}\right)^{\nu(n,k,\alpha)}$$

where

$$\nu(n,k,\alpha) = 2n + k + \alpha + 1/2, \qquad K_\alpha = \Gamma(\alpha+1)/[2^{\alpha+1/2}\Gamma(\alpha+3/2)].$$

Thus, since $\gamma = \alpha + 1/2$ when $\alpha = \beta$, one has:

$$\int_0^\infty p_s(x,y)\, s^{-3/2} e^{-\gamma^2 s}\, e^{-\frac{t^2}{4s}} ds = \frac{\sqrt{2\pi}\Gamma(\alpha+1)}{2^\alpha \Gamma(\alpha+3/2)} \frac{\tanh(t)}{t} W(y)$$
$$\sum_{n,k\geq 0} \frac{\Gamma(\nu(n,k,\alpha)+1)(xy)^k}{k!\, n!\, \Gamma(\alpha+n+1)} \left[\frac{(1-x^2)(1-y^2)}{4}\right]^n \left(\frac{1}{\cosh t}\right)^{\nu(n,k,\alpha)}$$

or equivalently:

$$\int_0^\infty p_{1/2s}(x,y) e^{-\frac{\gamma^2}{2s}} e^{-\frac{t^2}{2}s} \frac{ds}{\sqrt{s}} = \frac{\sqrt{\pi}\Gamma(\alpha+1)}{2^\alpha \Gamma(\alpha+3/2)} \frac{\tanh(t)}{t} W(y)$$
$$\sum_{n,k\geq 0} \frac{\Gamma(\nu(n,k,\alpha)+1)(xy)^k}{k!\, n!\, \Gamma(\alpha+n+1)} \left[\frac{(1-x^2)(1-y^2)}{4}\right]^n \left(\frac{1}{\cosh t}\right)^{\nu(n,k,\alpha)}.$$

Using (5), (6), f_{C_h} et f_{T_1} (we take $t^2/2$ instead of $t^2/8$), the density is written:

$$p_{1/2s}(x,y) = \frac{\sqrt{\pi s}\Gamma(\alpha+1)}{2^\alpha \Gamma(\alpha+3/2)} W(y) e^{\frac{\gamma^2}{2s}}$$
$$\sum_{n,k\geq 0} \frac{\Gamma(\nu(n,k,\alpha)+1)}{k!n!\Gamma(\alpha+n+1)} \left(\frac{xy}{2}\right)^k \left[\frac{(1-x^2)(1-y^2)}{4}\right]^n f_{T_1} \star f_{C_{\nu(n,k,\alpha)}}(s).$$

Finally

$$p_t(x,y) = \sqrt{\pi} K_\alpha \frac{e^{\gamma^2 t}}{\sqrt{t}} W(y) \sum_{n,k\geq 0} \frac{\Gamma(\nu(n,k,\alpha)+1)(xy)^k}{k!n!\Gamma(\alpha+n+1)}$$
$$\left[\frac{(1-x^2)(1-y^2)}{4}\right]^n f_{T_1} \star f_{C_{\nu(n,k,\alpha)}}\left(\frac{1}{2t}\right). \tag{7}$$

2. Application to statistics for diffusions processes

2.1. Some properties of the Jacobi process. Usually in probability theory, the Jacobi process is defined on $[-1,1]$ as the unique strong solution of the SDE :

$$dY_t = \sqrt{1-Y_t^2}\, dW_t + (bY_t + c)dt.$$

It is straightforward that $(Y_t)_{t\geq 0} \overset{\mathscr{L}}{=} (X_{t/2})_{t\geq 0}$ where X is the Jacobi process already defined in section 1 with $p = 2b$, $q = 2c$. Using the variable change $y \mapsto (y+1)/2$, the equation above transforms to $(t \to 4t)$:

$$\begin{aligned} dJ_t &= 2\sqrt{J_t(1-J_t)}dW_t + [2(c-b) + 4bJ_t]\,dt \\ &= 2\sqrt{J_t(1-J_t)}dW_t + [d - (d+d')J_t]\,dt \end{aligned}$$

where $d = 2(c-b) = q-p = 2(\beta+1)$ and $d' = -2(c+b) = -(p+q) = 2(\alpha+1)$, which is the Jacobi process of parameters (d, d') already considered by Warren and Yor ([21]). Moreover, the authors provide the following skew-product : let Z_1, Z_2 be two independent Bessel processes of dimensions d, d' and starting from z, z' respectively. Then :

$$\left(\frac{Z_1^2(t)}{Z_1^2(t)+Z_2^2(t)}\right)_{t\geq 0} \overset{\mathscr{L}}{=} (J_{A_t})_{t\geq 0}\,, \quad A_t := \int_0^t \frac{ds}{Z_1^2(s)+Z_2^2(s)}, \quad J_0 = \frac{z}{z+z'}.$$

Using well known properties of squared Bessel processes (see [20]), one deduce that if $d \geq 2\,(\beta \geq 0)$ and $z > 0$, then $J_t > 0$ almost surely for all $t > 0$. Since $1 - J$ is still a Jacobi process of parameters (d', d), then for $d' \geq 2$, $(\alpha \geq 0)$ and $z' > 0$, $J_t < 1$ almost surely for all $t > 0$. The extension of these results to the matrix Jacobi process is established in [8] (Theorem 3.3.2, p.36). Since 0 is a reflecting boundary for Z_1, Z_2 when $0 < d,\, d' < 2\,(-1 < \alpha, \beta < 0)$, then both 0 and 1 are reflecting boundaries for J.

2.2. **LDP in the ultraspherical case.** Let us consider the following SDE corresponding to the ultraspherical Jacobi process:

$$\begin{cases} dY_t = \sqrt{1-Y_t^2}dW_t + bY_t dt \\ Y_0 = y_0 \in]-1,1[. \end{cases} \tag{8}$$

Let $Q^b_{y_0}$ be the law of $(Y_t,\, t \geq 0)$ on the canonical filtered probability space $(\Omega, (\mathcal{F}_t), \mathcal{F})$ where Ω is the space of $]-1,1[$–valued functions. The parameter b is such that $b \leq -1$ (or $\alpha \geq 0$), so that $-1 < Y_t < 1$ for all $t > 0$. The maximum likelihood estimate of b based on the observation of a single trajectory $(Y_s,\, 0 \leq s \leq t)$ under Q^b_0 (see Overbeck [18] for more details) is given by

$$\hat{b}_t = \frac{\displaystyle\int_0^t \frac{Y_s}{1-Y_s^2}\,dY_s}{\displaystyle\int_0^t \frac{Y_s^2}{1-Y_s^2}\,ds}. \tag{9}$$

The main result of this section is the following Theorem.

Theorem 1. *When $b \leq -1$, the family $\{\hat{b}_t\}_t$ satisfies a LDP with speed t and good rate function*

$$J_b(x) = \begin{cases} -\dfrac{1}{4}\dfrac{(x-b)^2}{x+1} & if \quad x \leq x_0 \\ x + 2 + \sqrt{(b-x)^2 + 4(x+1)} & if \quad x > x_0 > b \end{cases} \tag{10}$$

where x_0 *is the unique solution of the equation* $(b-x)^2 = 4x(x+1) = 0$, $x < -1$.

Proof of Theorem 1 : we follow the scheme of Theorem 3.1 in [24]. Set :

$$S_{t,x} := \int_0^t \frac{Y_s}{1-Y_s^2}\, dY_s - x\int_0^t \frac{Y_s^2}{1-Y_s^2}\, ds,$$

so that for $x > b$ (resp. $x < b$), $P(\hat{b}_t \geq x) = P(S_{t,x} \geq 0)$ (resp. $P(\hat{b}_t \leq x) = P(S_{t,x} \leq 0)$). Therefore, to derive a large deviation principle on $\{\hat{b}_t\}$, we seek a LDP result for $S_{t,x}/t$ at 0. Let us compute the normalized cumulant generating function $\Lambda_{t,x}$ of $S_{t,x}$:

$$\Lambda_{t,x}(\phi) = \frac{1}{t}\log Q_0^b(e^{\phi S_{t,x}}). \tag{11}$$

¿From Girsanov formula, the generalized densities are given by

$$\frac{dQ_0^b}{dQ_0^{b'}}\Big|_{\mathcal{F}_t} = \exp\left\{(b-b')\int_0^t \frac{Y_s}{1-Y_s^2}\, dY_s - \frac{1}{2}(b^2-b'^2)\int_0^t \frac{Y_s^2}{1-Y_s^2}\, ds\right\}. \tag{12}$$

Let

$$F(Y_t) = -\frac{1}{2}\log(1-Y_t^2)\,.$$

From Itô formula,

$$F(Y_t) = \int_0^t \frac{Y_s}{1-Y_s^2}\, dY_s + \frac{1}{2}\int_0^t \frac{1+Y_s^2}{1-Y_s^2}\, ds = \int_0^t \frac{Y_s}{1-Y_s^2}\, dY_s + \int_0^t \frac{1}{1-Y_s^2}\, ds - \frac{t}{2}.$$

Let us denote by

$$\mathcal{D}_1(x) = \{\phi\, : (b+1)^2 + 2\phi(x+1) \geq 0\}\,.$$

For any $\phi \in \mathcal{D}_1(x)$, we can define $b(\phi,x) = -1 - \sqrt{(b+1)^2 + 2\phi(x+1)}$. With the change of probability defined by (12) taking $b' = b(\phi,x)$, the stochastic integrals simplify to (see [24] p. 125 for the details):

$$\Lambda_t(\phi,x) = \frac{1}{t}\log Q_0^{b(\phi,x)}(\exp(\{\phi + b - b(\phi,x))[F(Y_t) - t/2]\}). \tag{13}$$

When starting from $y_0 = 0$, (7) reads ($t \to t/2$) :

$$\tilde{p}_t(0,y) = \sqrt{2\pi}K_\alpha \frac{e^{\gamma^2 t/2}}{\sqrt{t}} \sum_{n\geq 0} \frac{\Gamma(2n+\alpha+\frac{3}{2})}{4^n n!\Gamma(n+\alpha+1)}(1-y^2)^{n+\alpha} f_{T_1} \star f_{C_{2n+\gamma}}(1/t),$$

where $p = -2(\alpha+1) = 2b \leq -2$ and $\gamma = -(p+1)/2 = \alpha + 1/2$. Denote by

$$\mathcal{D}(x) = \{\phi \in \mathcal{D}_1(x) : G(\phi,x) = b + b(\phi,x) + \phi < 0\}\,. \tag{14}$$

For any $\phi \in \mathcal{D}(x)$, the expectation (13) is finite and a simple computation gives :

$$\begin{aligned}\Lambda_t(\phi,x) &= -\frac{\phi + b - b(\phi,x)}{2} + \frac{1}{t}\log Q_0^{b(\phi,x)}((1-Y_t^2)^{-(\phi+b-b(\phi,x))/2}) \\ &= \Lambda(\phi,x) + \frac{1}{t}\log\frac{\sqrt{2\pi}K_{\alpha(\phi,x)}R_t(\phi,x)}{\sqrt{t}},\end{aligned}$$

where

$$\begin{aligned} R_t(\phi,x) &= \sum_{n\geq 0} \frac{\Gamma(2n-b(\phi,x)+1/2)}{4^n n!\Gamma(n-b(\phi,x))} B\left(n-\frac{\phi+b+b(\phi,x)}{2},\frac{1}{2}\right) e^{\gamma^2 t/2} f_{T_1}\star f_{C_{2n+\gamma}}(\frac{1}{t})\,, \\ \alpha(\phi,x) &= -b(\phi,x)-1 \end{aligned}$$

and B stands for the Beta function. With regard to (1), one has for $\phi\in\mathcal{D}(x)$:

$$\lim_{t\to\infty} Q_0^{b(\phi,x)}((1-Y_t^2)^{-(\phi+b-b(\phi,x))/2}) = C_{b,\phi,x}\int_{-1}^{1}(1-y^2)^{-[\phi+b+b(\phi,x)]/2-1}\,dy\ <\infty$$

by dominated convergence Theorem. Hence $\Lambda_t\to\Lambda$ as $t\to\infty$. The following lemma, which proof is postponed to the appendix, details the domain $\mathcal{D}(x)$ (see (14)) of Λ_t:

Lemma 1. *Denote by*

$$\phi_0(x) = -\frac{(b+1)^2}{2(x+1)}\,.$$

i)If $x<(b^2+3)/2(b-1)$: then $\mathcal{D}=(-\infty,\phi_0(x))$.

ii) If $(b^2+3)/2(b-1)<x<-1$: then $\mathcal{D}(x)=(-\infty,\phi_1(x))$ where $\phi_1(x)$ is solution of $G(\phi,x)=0$.

iii) If $x>-1$: then $\mathcal{D}(x)=(\phi_0(x),\phi_1(x))$.

In case i) of Lemma above, Λ is steep, i.e. its gradient is infinite at the boundary of the domain (for a precise definition, see [10]). It achieves its unique minimum in $\phi_m(x)$ solution of

$$\frac{\partial\Lambda}{\partial\phi}(\phi,x)=0\,,$$

i.e. $b(\phi(x),x)=x$. It is easy to see that

$$\phi_m(x)=\frac{x+1}{2}-\frac{(b+1)^2}{2(x+1)}<\phi_0(x)\,.$$

Hence, Gärtner-Ellis Theorem gives for $x<b<(b^2+3)/2(b-1)$,

$$\lim_{t\to\infty}\frac{1}{t}\log P(\hat{b}_t\leq x)=\lim_{t\to\infty}\frac{1}{t}\log P(S_{t,x}\leq 0)=\inf_{\phi\in]\infty,\phi_0(x)]}\Lambda(\phi,x)=\Lambda(\phi_m(x),x)=-\frac{1}{4}\frac{(x-b)^2}{x+1}.$$

If $b<x<(b^2+3)/2(b-1)$, notice that $\phi_m(x)>0$ and

$$\lim_{t\to\infty}\frac{1}{t}\log P(\hat{b}_t\geq x)=\lim_{t\to\infty}\frac{1}{t}\log P(S_{t,x}\geq 0)=\inf_{\phi\in(0,\phi_0(x)]}\Lambda(\phi,x)=\Lambda(\phi_m(x),x)=-\frac{1}{4}\frac{(x-b)^2}{x+1}.$$

In cases ii) and iii) of Lemma 1, Λ is not steep. Nevertheless, if the infimum of Λ is reached in $\overset{\circ}{\mathcal{D}}(x)$, we can follow the scheme of Gärtner–Ellis theorem for the change of probability in the infimum bound. This infimum is reached if and only if

$$\frac{\partial\Lambda}{\partial\phi}(\phi_1(x),x)>0\,,\text{i.e. if }\phi_m(x)<\phi_1(x). \tag{15}$$

In case $x+1>0$, we know (see proof of Lemma 1) that $\phi_1(x) < \phi_m(x)$. If $x+1<0$, we check the sign of $G(\phi_m(x),x)$. We get the following dichotomy : Let x_0 denote the unique solution of $g(x) := 4x(x+1)-(b-x)^2 = 0,\ x<-1$. Since g is decreasing on $]-\infty,-1]$ and $g(b^2+3/(2(b-1)) = (3/4)(b+1)^2 > 0 = g(x_0)$, then $x_0 > (b^2+3)/[2(b-1)]$.
• if $(b^2+3)/2(b-1) < x < x_0 < -1$, the derivative $\partial\Lambda/\partial\phi(\phi_1(x),x) > 0$, Λ achieves its minimum on $\phi_m(x)$ and

$$\lim_{t\to\infty}\frac{1}{t}\log P(\hat{b}_t \geq x) = \Lambda(\phi_m(x),x) = -\frac{(x-b)^2}{4(x+1)}\,.$$

• if $x_0 < x < -1$ or $x > -1$, then $\partial\Lambda/\partial\phi(\phi_1(x),x) < 0$. We apply Theorem 2 of the appendix, which is due to Zani [24]. Let us verify that the assumptions are satisfied, and more precisely that Λ_t can take the form (18). Indeed, the only singularity $\phi_1(x)$ of R_t comes from $B\left(n-[\phi+b+b(\phi,x)]/2,1/2\right)$ when $n=0$, and more precisely, from $\Gamma(-[\phi+b+b(\phi,x)]/2)$. We can write

$$\Lambda_t(\phi,x) = \Lambda(\phi,x) + \frac{1}{t}\log\Gamma\left(-\frac{\phi+b+b(\phi,x)}{2}\right) + \frac{1}{t}\log\frac{\sqrt{2\pi}K_{\alpha(\phi,x)}\tilde{R}_t(\phi,x)}{\sqrt{t}}\,, \tag{16}$$

where

$$\tilde{R}_t(\phi,x) = R_t(\phi,x)/\Gamma(-[\phi+b+b(\phi,x)]/2). \tag{17}$$

Now

$$\forall n \geq 0, \quad B\left(n-\frac{\phi+b+b(\phi,x)}{2},\frac{1}{2}\right)/\Gamma\left(-\frac{\phi+b+b(\phi,x)}{2}\right)$$

is analytic on some neighbourhood of $\phi_1(x)$. Besides,

$$\lim_{\phi\to\phi_1(x)\,,\phi<\phi_1(x)}\frac{b+\phi+b(\phi,x)}{\phi-\phi_1(x)} = c > 0\,,$$

and since $\lim_{\rho\to 0^+}\rho\Gamma(\rho) = 1$, then $\phi_1(x)$ is a pole of order one of $\Gamma\left(\phi+b+b(\phi,x)/2\right)$ and one writes:

$$\frac{1}{t}\log\Gamma\left(-\frac{\phi+b+b(\phi,x)}{2}\right) = -\frac{\log(\phi_1(x)-\phi)}{t} + \frac{h(\phi)}{t}\,.$$

The function h is analytic on $\mathcal{D}(x)$ and can be extended to an analytic function on $]\phi_1(x)-\xi,\phi_1(x)+\xi[$ for some positive ξ.
Finally, to satisfy **Assumption 1** of the appendix, we focus on $\tilde{R}_t(\phi,x)/\sqrt{t}$ and show that it is bounded uniformly as $t\to\infty$. To proceed, we shall prove that this ratio is bounded from above and below away from 0. Setting $A_n(t) := e^{\gamma^2 t/2} f_{T_1} \star f_{C_{2n+\gamma}}(1/t)$,

one has :

$$\begin{aligned}
\frac{A_n(t)}{\sqrt{t}} &\leq \frac{e^{\gamma^2 t/2}}{\sqrt{t}} \sum_{k,l\geq 0} U_{k,n} \int_0^{1/t} \exp -\frac{1}{2}\left[\frac{(2n+2k+\gamma)^2}{s} + \pi^2(l+\frac{1}{2})^2(\frac{1}{t}-s)\right]\frac{ds}{s^{3/2}} \\
&= \frac{e^{\gamma^2 t/2}}{\sqrt{t}} \sum_{k,l\geq 0} U_{k,n} \int_t^{\infty} \exp -\frac{1}{2}\left[(2n+2k+\gamma)^2 s + \pi^2(l+\frac{1}{2})^2(\frac{s-t}{ts})\right]\frac{ds}{\sqrt{s}} \\
&< e^{-2n^2} \sum_{k,l\geq 0} U_{k,n} e^{-2k^2} \int_t^{\infty} \exp -\frac{1}{2}\left[(2n+2k+\gamma)^2 (s-t) + \pi^2(l+\frac{1}{2})^2(\frac{s-t}{ts})\right]\frac{ds}{\sqrt{ts}} \\
&= e^{-2n^2} \sum_{k,l\geq 0} U_{k,n} e^{-2k^2} \int_0^{\infty} \exp -\frac{1}{2}\left[(2n+2k+\gamma)^2 s + \pi^2 l^2(\frac{s}{t(t+s)})\right]\frac{ds}{\sqrt{t(t+s)}}
\end{aligned}$$

with

$$U_{k,n} = \frac{\Gamma(2n+k+\gamma)2^{2n+\gamma}(2n+2k+\gamma)}{k!\Gamma(2n+\gamma)}.$$

Let $\Theta(x) = \sum_{l\in\mathbb{Z}} e^{-\pi l^2 x} = 1 + 2\sum_{l\geq 1} e^{-\pi l^2 x}$ denote the Jacobi Theta function ([5]). Then

$$\frac{A_n(t)}{\sqrt{t}} < e^{-2n^2}\left[\sum_{k\geq 0} U_{k,n} e^{-2k^2} \int_0^{\infty} \exp -\left[\frac{(2n+2k+\gamma)^2 s}{2}\right]\Theta\left(\frac{\pi s}{2t(t+s)}\right)\frac{ds}{\sqrt{t(t+s)}} + C(n,t)\right]$$

where

$$C(n,t) = \frac{1}{2\sqrt{t}} \sum_{k,l\geq 0} U_{k,n} e^{-2k^2} \int_0^{\infty} \exp -\left[\frac{(2n+2k+\gamma)^2 s}{2}\right]\frac{ds}{\sqrt{t+s}}.$$

Recall that $\Theta(x) = (1/\sqrt{x})\Theta(1/x)$ ([5]), which yields :

$$\frac{A_n(t)}{\sqrt{t}} < e^{-2n^2} \sum_{k\geq 0} U_{k,n} e^{-2k^2} \int_0^{\infty} \exp -\left[\frac{(2n+2k+\gamma)^2 s}{2}\right]\Theta\left(\frac{2t(t+s)}{\pi s}\right)\frac{ds}{\sqrt{s}} + \frac{C(n)}{2\sqrt{t}}$$

where

$$C(n) = e^{-2n^2} \sum_{k,l\geq 0} U_{k,n} e^{-2k^2} \int_0^{\infty} \exp -\left[\frac{(2n+2k+\gamma)^2 s}{2}\right]\frac{ds}{\sqrt{s}}.$$

Since $e^{-l^2 z} < e^{-lz}$, then $\Theta(z) \leq 3$ for $z > 1$. Hence, as $2t/\pi \leq 2t(t+s)/(\pi s)$, then for t large enough:

$$\frac{A_n(t)}{\sqrt{t}} < 3e^{-2n^2} \sum_{k\geq 0} U_{k,n} e^{-2k^2} \int_0^{\infty} \exp -\left[\frac{(2n+2k+\gamma)^2 s}{2}\right]\frac{ds}{\sqrt{s}} + C(n) < 4C(n).$$

The upper bound follows since $\sum_n C(n) < \infty$. Besides,

$$\frac{\tilde{R}_t(\phi,x)}{\sqrt{t}} > \frac{\sqrt{\pi}\Gamma(1/2-b(\phi,x))}{\Gamma(-b(\phi,x))\Gamma\{[1-(\phi+b+b(\phi,x)]/2\}}\frac{A_0(t)}{\sqrt{t}}$$
$$= C(b,\phi,x)\sum_{k,l\geq 0}(-1)^k V_k\int_0^\infty \exp -\frac{1}{2}\left[(2k+\gamma)^2 s+\pi^2(l+\frac{1}{2})^2(\frac{s}{t(t+s)})\right]\frac{ds}{\sqrt{t(t+s)}}$$

where $V_k(t) := U_{k,0}e^{-2k(k+\gamma)t}$. One may choose t large enough independent of k such that $V_k(t) \geq V_{k+1}(t)$ for all $k \geq 0$. In fact, such t satisfies:

$$e^{2(2k+\gamma+1)t} \geq e^{2t} \geq \sup_{k\geq 0}\frac{U_{k+1,0}}{U_{k,0}} = \sup_{k\geq 0}\frac{(k+\gamma)(2k+\gamma+2)}{(k+1)(2k+\gamma)}$$

Then:

$$\frac{\tilde{R}_t}{\sqrt{t}} > C(b,\phi,x)[V_0(t)-V_1(t)]\sum_{l\geq 0}\int_0^\infty \exp -\frac{1}{2}\left[\gamma^2 s+\pi^2(l+\frac{1}{2})^2(\frac{s}{t(t+s)})\right]\frac{ds}{\sqrt{t(t+s)}}$$
$$> C(b,\phi,x)[\gamma 2^\gamma - V_1(t)]\sum_{l\geq 0}\int_0^\infty \exp -\frac{1}{2}\left[\gamma^2 s+\pi^2(l+1)^2(\frac{s}{t(t+s)})\right]\frac{ds}{\sqrt{t(t+s)}}$$
$$= \frac{C(b,\phi,x)}{2}[\gamma 2^\gamma - V_1(t)]\left\{\int_0^\infty e^{-\gamma^2 s/2}\Theta\left(\frac{\pi s}{2t(t+s)}\right)\frac{ds}{\sqrt{t(t+s)}} - C(t)\right\}$$

where

$$C(t) = \frac{1}{2\sqrt{t}}\int_0^\infty e^{-\gamma^2 s/2}\frac{ds}{\sqrt{(t+s)}} < c\int_0^\infty e^{-\gamma^2 s/2}\frac{ds}{\sqrt{s}}, \quad c < \sqrt{\frac{2}{\pi}}.$$

for t large enough. Following the same scheme as for the upper bound, one gets:

$$\frac{\tilde{R}_t}{\sqrt{t}} > \frac{C(b,\phi,x)}{2}\gamma 2^\gamma\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\gamma^2 s/2}\Theta\left(\frac{2t(t+s)}{\pi s}\right)\frac{ds}{\sqrt{s}} - C(t)\right\}$$
$$> \frac{C(b,\phi,x)}{2}\gamma 2^\gamma\left(\sqrt{\frac{2}{\pi}} - c\right)\int_0^\infty e^{-\gamma^2 s/2}\frac{ds}{\sqrt{s}} > 0. \qquad \square$$

As a result,

$$\lim_{t\to\infty}\frac{1}{t}\log P(\hat{b}_t \geq x) = \Lambda(\phi_1(x),x) = -(x+2+\sqrt{(b-x)^2+4(x+1)}),$$

which ends the proof of Theorem 1. ■

2.3. **Jacobi and squared Bessel processes duality.** By Itô's formula and Lévy criterion, one claims that $(Z_t = Y_t^2)_{t\geq 0}$ is a $[0,1]$-valued Jacobi process of parameters $d = 1$, $d' = -2b \geq 2$. Indeed:

$$dZ_t := d(Y_t^2) = 2Y_t dY_t + \langle Y\rangle_t = 2Y_t\sqrt{1-Y_t^2}dW_t + [(2b-1)Y_t^2+1]dt$$
$$= 2\sqrt{Z_t(1-Z_t)}\mathrm{sgn}(Y_t)dW_t + [(2b-1)Z_t+1]dt$$
$$= 2\sqrt{Z_t(1-Z_t)}dB_t + [(2b-1)Z_t+1]dt.$$

Using the skew product previously stated, there exists R, a squared Bessel process of dimension $d' = 2(\nu+1) = -2b$ and starting from $R_0 = r$ so that:

$$\hat{\nu}_t := -\hat{b}_t - 1 = \frac{\log(1-Y_t^2)+t}{2\int_0^t \frac{Y_s^2}{1-Y_s^2}ds}$$

is another estimator of ν based on a Jacobi trajectory observed till time t. Set $t = \log u$, then

$$\hat{\nu}_{\log u} = \frac{\log[u(1-Y_{\log u}^2)]}{2\int_0^{\log u} \frac{Y_s^2}{1-Y_s^2}ds} = \frac{\log[u(1-Y_{\log u}^2)]}{2\int_1^u \frac{Y_{\log s}^2}{s(1-Y_{\log s}^2)}ds}$$

and $\{\hat{\nu}_{\log u}\}_u$ satisfies a LDP with speed $\log u$ and rate function $J_{-(\nu+1)}(-(x+1))$.

When starting at $R_0 = 1$, the MLE of ν based on a Bessel trajectory is given by (cf [24], p. 132):

$$\hat{\nu}_t^1 = \frac{\int_0^t \frac{dX_s}{X_s} - 2\int_0^t \frac{ds}{X_s}}{2\int_0^t \frac{ds}{X_s}} = \frac{\log(X_t)}{2\int_0^t \frac{ds}{X_s}}$$

with associated rate function :

$$I_\nu(x) = \begin{cases} \frac{(x-\nu)^2}{4x} & \text{if} \quad x \geq x_1 \quad := \frac{-(\nu+2)+2\sqrt{\nu^2+\nu+1}}{3} \\ 1 - x + \sqrt{(\nu-x)^2 - 4x} & \text{if} \quad x < x_1. \end{cases}$$

A glance at both rate functions gives $I_\nu(x) = J_{-(\nu+1)}(-(x+1))$ and $x_0 = -(x_1+1)$.

3. Appendix

3.1. **A large deviations principle in a non steep case.** Let $\{Y_t\}_{t\geq 0}$ be a family of real random variables defined on $(\Omega, \mathcal{F}, P)$, and denote by μ_t the distribution of Y_t. Suppose $-\infty < m_t := EY_t < 0$. We look for large deviations bounds for $P(Y_t \geq y)$. Let Λ_t be the n.c.g.f. of Y_t:

$$\Lambda_t(\phi) = \frac{1}{t}\log\, E(\exp\{\phi t Y_t\}),$$

and denote by D_t the domain of Λ_t. We assume that there exists $0 < \phi_1 < \infty$ such that for any t

$$\sup\{\phi : \phi \in D_t\} = \phi_1$$

and $[0, \phi_1) \subset D_t$. We assume also that for $\phi \in D_t$

Assumption 1.

$$\Lambda_t(\phi) = \Lambda(\phi) - \frac{\alpha}{t}\log(\phi_1 - \phi) + \frac{R_t(\phi)}{t} \tag{18}$$

where

- $\alpha > 0$
- Λ is analytic on $(0, \phi_1)$, convex, with finite limits at endpoints, such that $\Lambda'(0) < 0$, $\Lambda'(\phi_1) < \infty$, and $\Lambda''(\phi_1) > 0$.
- R_t is analytic on $(0, \phi_1)$ and admits an analytic extension on a strip $D_\beta^\gamma = (\phi_1 - \beta, \phi_1 + \beta) \times (-\gamma, \gamma)$, where β and γ are independent of t.
- $R_t(\phi)$ converges as $t \to \infty$ to some $R(\phi)$ uniformly on any compact of D_β^γ.

Theorem 2. *Under 1*
For any $\Lambda'(0) < y < \Lambda'(\phi_1)$,

$$\lim_{t\to+\infty} \frac{1}{t} \log P(Y_t \geq y) = -\sup_{\phi\in(0,\phi_1)} \{y\phi - \Lambda(\phi)\}\,. \tag{19}$$

For any $y \geq \Lambda'(\phi_1)$,

$$\lim_{t\to+\infty} \frac{1}{t} \log P(Y_t \geq y) = -y\phi_1 + \Lambda(\phi_1)\,. \tag{20}$$

The rate function is continuously differentiable with a linear part.

3.2. **Proof of Lemma 1 :** Note first that $(b^2+3)(2(b-1)) < -1$ if $b < -1$ and that the condition $\phi \in \mathcal{D}_1(x) \Rightarrow \phi \geq \phi_0(x)$ if $x > -1$ and $\phi \leq \phi_0(x)$ if $x < -1$. To examine the behaviour of G, we compute

$$\frac{\partial G}{\partial \phi}(\phi, x) = 1 - \frac{x+1}{\sqrt{(b+1)^2 + 2\phi(x+1)}}\,.$$

• If $x+1 < 0$, $\dfrac{\partial G}{\partial \phi} > 0$ and $G(\cdot, x)$ is increasing. Then we see easily that $G(\phi_0(x), x) < 0$ iff $x < (b^2+3)(2(b-1))$, which determines cases i) and ii).
• If $x+1 > 0$, $\phi \to \dfrac{\partial G}{\partial \phi}$ is increasing hence there exists $\tilde{\phi}(x)$ such that $\dfrac{\partial G}{\partial \phi}(\tilde{\phi}(x), x) = 0$. We compute

$$\tilde{\phi}(x) = \frac{x+1}{2} - \frac{(b+1)^2}{2(x+1)} = \phi_m(x)\,.$$

We see that $G(\tilde{\phi}(x), x) < 0$, and there exists $\phi_1(x) < \tilde{\phi}(x)$ such that $G(\phi_1(x), x) = 0$, and $\mathcal{D}(x) = (\phi_0(x), \phi_1(x))$. □

References

[1] *D. Applebaum.* Quantum Independent Increment Processes I : From Classical Probability to Quantum Stochastic Calculus. *Springer, Lecture Notes in Mathematics.* Vol. 1865, 2005, 1-99.
[2] *G. E. Andrews, R. Askey, R. Roy.* Special functions. *Cambridge University Press.* 1999.
[3] *D. Bakry, O. Mazet.* Characterization of Markov Semi-groups on $\mathbb{R}$ Associated to Some Families of Orthogonal Polynomials. *Sem. Proba. XXXVI. Lecture Notes in Maths. Springer.* Vol. 1832, 2002. 60-80.
[4] *T. H. .Baker, P. J. Forrester..* The Calogero- Sutherland Model and generalized Classical Polynomials. *Comm. Math. Phy.* 188, 1997, 175-216.
[5] *P. Biane, J. Pitman, M. Yor.* Probability Laws Related To The Jacobi Theta and Riemann Zeta Functions, and Brownian Excursions. *Bull. Amer. Soc.* 38, no. 4, 2001, 435-465.
[6] *Yu.A.Brychkov, O.I.Marichev, A.P.Prudnikov.* Integrals and Series, Vol. 2: special functions. *Gordon and Breach science publishers.*
[7] *S. Bochner.* Sturm-Liouville and heat equations whose eigenfunctions are ultraspherical polynomials or associated Bessel functions. *Proc. Conf. Diff. Eq.*, 1955, 23-48.
[8] *Y. Doumerc.* Matrix Jacobi Process. *Ph. D. Thesis.* Paul Sabatier University. 2005
[9] *F. Delbaen, H. Shirakawa.* An interest rate model with upper and lower bound. *Asia-Pacific Financial Markets*, **9**, 2002, 191–209.
[10] *A. Dembo, O. Zeitouni.* Large Deviations Techniques and Applications. *Springer.* 1998.
[11] *S. Ethier, T. G. Kurtz.* Markov Processes : Characterization and Convergence. *John Wiley and Sons, Inc.* 1986

[12] *H. Exton*. Multiple Hypergeometric Functions And Applications. *Ellis Horwood Limited*. 1976.
[13] *G. Gasper*. Banach algebras for Jacobi series and positivity of a Kernel. *Ann. Math.* **95**, 1972, 261-280.
[14] *S. P. Karlin, G. McGregor*. Classical diffusion processes and total positivity. *J. Math. Anal. Appl.* **1**, 1960, 163-183.
[15] *M. Lassalle*. Polynômes de Jacobi généralisés. *C. R. A. S. Paris*. Séries I **312**, 1991, 425-428.
[16] *N.N.Lebedev*. Special Functions And Their Applications. *Dover Publications, INC*. 1972.
[17] *W. Magnus, F. Oberhettinger, R. P. Soni*. Formulas And Theorems for the Special Functions of Mathematical Physics. *Springer-Verlag New York, Inc. 1996.*
[18] *L. Overbeck*. Estimation for continuous branching processes. *Scand. Journ. of Stat.* **25**, 1998, 111-126.
[19] *J. Pitman, M. Yor*. Infinitely Divisible Laws Associated With Hyperbolic Functions. *Canad. J. Math.* **55**, no. 2, 2003, 292-330.
[20] *D. Revuz, M. Yor*. Continuous Martingales And Brownian Motion, 3^{rd} ed, Springer, 1999
[21] *J. Warren, M. Yor*. The Brownian Burglar : Conditionning Brownian motion by its local time process. *Sém. Proba. Stras. XXXII.*, 1998, 328-342.
[22] *E. Wong*. The construction of a class of stationnary Markov. *Proceedings. The* 16^{th} *Symposium. Applied Math*. AMS. Providence. RI. 1964. 264-276.
[23] *M. Yor*. Loi de l'indice du lacet brownien et distribution de Hartman-Watson. *P.T.R.F*, **53**, no. 1, 1980, 71-95.
[24] *M. Zani*. Large deviations for squared radial Ornestein-Uhlenbeck processes. *Stoch. Proc. App.* **102**, no. 1, 2002, 25-42.
[25] *M. Zani*. Grandes déviations pour des fonctionnelles issues de la statistique des processus. *Ph. D. Thesis*. Orsay University. 2000.